Stringing the Harpsichord – Some Physical Considerations

Klaus Scholl

"There are fine books in plenty about the history of particular musical instruments, lavishly illustrated with photographs and drawings, but there is virtually nothing outside the scientific journal literature which attempts to come to grips with the subject on a quantitative basis." This statement in the preface of the the book *The Physics of Musical Instruments* by Neville H. Fletcher and Thomas D. Rossing [2] is also true for the literature on the harpsichord.

Especially, if one is looking for a rule by which the string diameter should be increased from treble to bass, physically valid considerations can only be found in the book mentioned above and in Fletcher's article *Analysis of the Design and Performance of Harpsichords* [1].

Fletcher/Rossing had in mind the reader "with a reasonable grasp of physics and who is not frightened by a little mathematics" [2]. Since such a reader is rare among harpsichord makers, basic physical terms are explained and the mathematics are much simplified in this article.

The aim of this article is to introduce some practical stringing rules that are based on physical concepts, with due consideration of historical stringing lists. For readers with some knowledge of physics the underlying standard linear theory (neglecting nonlinear behavior of the vibrating string) is briefly summarized in the annexes.

Readers expecting a simple formula to calculate the string diameters will be disappointed. There is no such simple rule, and there will probably never be one considering the acoustical complexity of the harpsichord and the different concepts of traditional harpsichord making.

Nevertheless some calculations are recommended and will be demonstrated in the article. Any cheap scientific pocket calculator will suffice to execute them. A computer spread sheet application will, however, be much more convenient.

Preliminaries

A mathematical understanding of some basic terms is indispensable for reading the main part of this article. They will be explained in this section.

Cents and frequencies

The reader will certainly know that 1200 cents make an octave and that a semitone in equal temperament measures 100 cents. However, many a reader will have problems with the mathematics of cents. Therefore, taking meantone temperament as an example the calculations with cents are demonstrated below.

Meantone temperament is based on pure major thirds. The ratio of the two frequencies of this interval is 5 : 4 and the corresponding cent value – a dimensionless number – is calculated as follows:

$$interval_{(cent)} = \frac{1200}{\lg 2} \times \lg \frac{5}{4} = 386.314 \text{ cents}$$

Three major thirds constitute an octave; however, three pure thirds are short of an octave by

$$1200 - (3 \times 386,314) = 41.06 \text{ cents}$$

In meantone temperament the cent value of a whole tone is just half the value of a pure major third, that is 193.16 cents. A whole tone is divided in two unequal semitones that differ by the 41.06 cents just calculated. So we arrive at semitone values of 76.05 cents and 117.11 cents and simply by adding the appropriate cent values of the intervals we can construct the full scale of the meantone temperament. An octave is made up by five small and seven large semitones, where e–f and b–c are large semitones and raising(♯)/lowering(♭) a note is done by a small semitone.

The intervals expressed in cents – we will use the abbreviation I – are converted into frequency ratios

	I (cent)	*ratio*	f (1/sec)
b'	193.16	1.1180 : 1	491.9
$b^\flat$	117.11	1.0700 : 1	470.8
a'	0	1.0000 : 1	440.0
$g^\sharp$	-117.11	0.9346 : 1	411.2
g'	-193.16	0.8944 : 1	394.5

Tab. 1: Example: cents and frequencies

by means of the following formula:

$$frequency\ ratio = 2^{\frac{I}{1200}}$$

that is we first divide the interval given in cents by 1200 and then use the result as the exponent of 2.

By multiplying the frequency ratios with the tuning frequency (generally in harpsichords a' is tuned at 440 Hz or 415 Hz or 392 Hz) the fundamental frequency of each tone can be calculated. An example is shown in Tab. 1.

Newton: the unit of force

In everyday language the kilogram (= 2.2046 lb) is used as a unit of both mass and weight, whereby weight is equivalent to force. To make the different meanings clear, we will write kilogram in the meaning of weight with a suffix: $\mathrm{kg_w}$. A $\mathrm{kg_w}$ is the weight of the mass of one kg on our planet. Since the acceleration of gravity on the earth is 9.80665 m/sec^2 we can write the following formula:

$$\mathrm{kg_w} = \frac{\mathrm{kg} \times 9.80665\ \mathrm{m}}{\mathrm{sec}^2}$$

On the moon and on any other planet the acceleration of gravity is different. Therefore, it is desirable to have a unit of force without that constant, and that is the unit Newton, abbreviated N and defined as follows:

$$\mathrm{N} = \frac{\mathrm{kg} \times \mathrm{m}}{\mathrm{sec}^2}$$

In the words of Webster's dictionary: "Newton: force which imparts to a mass of one kilogram an acceleration of one meter per second per second."

In the natural sciences the term N is used exclusively and it will, therefore also be used in this article. If you want $\mathrm{kg_w}$, divide the values given in N by 9.81.

Tensile stress

The term tensile stress is physically defined in the same way as pressure: force per unit of area; only the direction of the force is reversed.

For strings of musical instruments the tensile stress is specified in the dimension N per mm^2.

The tensile stress – abbreviated S – is an important component of Taylor's famous formula, which every harpsichord maker should know:

$$f = \frac{1}{2L}\sqrt{\frac{S}{\varrho}}$$

where f is the fundamental frequency of the tone, L the sounding length of the string and ϱ (the Greek letter rho) the wire density.

Strictly speaking, the formula is not exact, since it does not take the inharmonicity into account. However, with the thin strings of the harpsichord the inharmonicity may be neglected. Harpsichords are always tuned with exact octaves, regardless of inharmonicity. The situation is different with pianos, which are tuned with stretched octaves because of the considerable inharmonicity of the thick piano strings [2, 7].

However, it is not the frequency we want to know (we know it already, see subsection "Cents and frequencies"), but rather the tensile stress. Therefore, the above formula must be written in the following form:

$$S = 4\varrho(fL)^2$$

The formula looks easy, but there are some pitfalls hidden in it. Usually we work with different dimensions:

- The density ϱ is usually given in the dimension grams (g) per cm^3.
- In this article the sounding length L of the string will always be given in mm.

Doing so we get the following dimensions in the above formula:

$$S = \frac{\mathrm{g} \times \mathrm{mm}^2}{\mathrm{cm}^3 \times \mathrm{sec}^2}$$

This must be converted into N per mm^2, an expression containing two different dimensions of the unit meter:

$$S = \frac{\mathrm{N}}{\mathrm{mm}^2} = \frac{\mathrm{kg} \times \mathrm{m}}{\mathrm{mm}^2 \times \mathrm{sec}^2}$$

The conversion is simple: divide the result by 10^9.

Tension

Physically the term tension denotes a force exerted by pulling. Thus, the unit for the tension is the same as the unit of force, namely Newton.

After the tensile stress has been calculated, the tension – abbreviated T – is given by the formula:

$$T = \frac{\pi d^2 S}{4}$$

where d is the diameter of the string.

It should be noted that the frequency depends on the tensile stress and not on the tension, although the latter is connected with the former by the above formula.

Decibels

Everyone who has ever worked with an audio tape recorder or one of its modern digital equivalents knows the level meters with their decibel scales. Decibels play an important role in any area of acoustics, and they will also be used in this article.

Like the cents just covered decibels are logarithmic expressions of ratios and thus dimensionless. Starting point is a measured or any other actual value, which is divided by a reference value. From the result the logarithm (base 10) is taken:

$$\lg \frac{actual\ value}{reference\ value}$$

For *sound pressures* and *voltages* and other *amplitudes* the logarithm of the ratio is multiplied by 20 and called level, which is expressed in decibels (abbreviated dB). An example: The actual voltage value is 1.55 volts and the reference value 0.775 volts.

$$Voltage\ level = 20 \times \lg \frac{1.55}{0.775} = 6\ \mathrm{dB}$$

The reference voltage of 0.775 volts happens to be the standardized reference level for professional analog audio equipment and the levels are called $\mathrm{dB_u}$.

If the reference value is chosen arbitrarily, the resulting levels are called *relative*; if it is standardized as in the example above, the levels are called *absolute*.

If the actual value is smaller than the reference value, the level becomes negative.

If we deal with acoustic or electric *power*, the result is multiplied by 10, since the power is proportional to the square of the sound pressure/voltage. An example which is related to the voltage example above: At an impedance of 600 ohms 1.55 volts deliver 4 milliwatts, whereas the reference voltage of 0.775 volts delivers a reference power of 1 milliwatt. What is the power level?

$$Power\ level = 10 \times \lg \frac{4}{1} = 6\ \mathrm{dB}$$

The *energy* stored in the string by plucking is proportional to the square of the deflection. Thus, the above formula with the multiplier 10 also applies to *energy levels* of the plucked string.

Plucking ratio and harmonic spectrum

The plucking ratio – abbreviated p – is calculated by dividing the distance – abbreviated P – from the plucking point to the *nearest* bridge by the sounding length L of the string:

$$p = \frac{P}{L}$$

Every harpsichord maker knows that the plucking ratio has an impact on the spectrum of overtones, but very few will be able to calculate that impact. Without bothering the reader with the theory[1] here is the formula to calculate the relative amplitude level A of the nth harmonic of the vibration transferred to the bridge by the string:

$$A_n[\mathrm{dB}] = 20 \lg \left| \frac{1}{n} \sin(np\pi) \right|$$

The attentive reader of the preceding subsection on decibels will note that the reference value seems

[1] The interested reader with advanced mathematical knowledge may read annex B.

to be missing in the formula. The answer is: The reference value is 1 and need not be written. It is the maximum possible value, reached only in the case of $n = 1$ and $p = 0.5$, where the level is 0 dB. Any other level is negative.[2]

If you use a scientific pocket calculator, switch from degree to radians and, before taking the logarithm, make the value positive, if it is negative (the bars in the formula mean absolute value), since you can't take the logarithm of a negative number. For the rest the formula should be straightforward to use.

For several plucking ratios Tab. 2 contains the relative amplitude levels in decibels of the first four harmonics (the first harmonic is the fundamental, the second the octave, the third 3 times the fundamental etc.). If you want to make a graph, use a logarithmic scale for the harmonics, that is the space between the 2nd and 4th harmonic should be the same as between the 1st and 2nd harmonic.

The plucking ratios in Tab. 2 have been chosen deliberately: 0.45 down to 0.33 are typical values in the treble, 0.09 is generally reached in the extreme bass, and 0.04 is a typical value for a lute register.

The explanation for the very low amplitude level of -39.6 dB at $p = 0.33$ is that the nth harmonic and its multiples are *theoretically*[3] missing, if a string is plucked at *exactly* one nth of its length. Such ratios are called *nodes*. In practice the case of hitting a node exactly is not very probable, but when the plucking point is coming close to a node, the theoretical amplitude level decreases rapidly.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$p = 0.45$	-0.1	-16.2	-10.5	-16.7
$p = 0.33$	-1.3	-7.2	-39.6	-13.5
$p = 0.21$	-4.3	-6.3	-10.3	-18.4
$p = 0.09$	-11.1	-11.4	-12.0	-12.9
$p = 0.04$	-18.0	-18.1	-18.2	-18.4

Tab. 2: Relative amplitude levels in decibels of the first 4 harmonics at different plucking ratios

The plucking point factor

The term plucking point factor has been coined by the author and will be explained in detail.

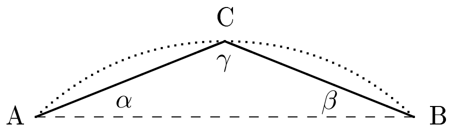
Every harpsichord maker knows that a string plucked near the nut produces a softer sound than a string plucked somewhat farther away. It is evident that the energy stored in the string by plucking somehow depends on the plucking ratio p, which has been defined in the preceding subsection.

A possible reason for the reduced loudness may be found in the amplitude spectrum of the overtones as discussed in the preceding subsection. However, the main factor for this phenomenon of decreased loudness is the *plucking point factor* φ (the Greek letter phi) which is mathematically defined as follows:

$$\varphi = p\,(1 - p)$$

The energy stored in the string by plucking as well as the initial vertical deflection of the string caused by the pluck are proportional to the plucking point factor.

The following figure may provide a better imagination of this relation.



In this drawing A and B are the string supports, point C is the summit of the initial deflection of the string caused by plucking. With the plucking force and the string tension kept constant the sum of the angles α and β will also be constant, if the deflection is very small like in the case of the harpsichord. The angle γ is therefore constant, too, and the summit of the deflection (point C) will move along a circle sector when the plucking point is changed. Thus, it is evident that with a constant plucking force the maximum deflection will be attained by plucking the string exactly in the middle between the bridges and that the deflection will decrease, if the plucking point is moved towards one of the bridges.

[2] The method of taking the maximum possible value as the reference is also used in the level meters – LED chains – of digital audio recording machines. The decibel markings of the LED's are all negative except the maximum value, which is zero.

[3] According to the standard linear theory. However, these modes are not actually missing, but rather generated in a nonlinear way. "They typically begin with near-zero amplitude, rise to a peak after a time of the order of 0.1 s, and then decay."[6]

	$\boldsymbol{L}$ (mm)	$\boldsymbol{P}$ (mm)	$\boldsymbol{I}$ (cent)	$\boldsymbol{S}$ (N/mm^2)	φ
c'''	178	70	1500	969	0.239
b''	189	71	1400	974	0.235
$b^{\flat}{}''$	200	72	1300	971	0.230
a''	212	73	1200	972	0.226
$g^{\sharp}{}''$	225	74	1100	976	0.221
g''	239	75	1000	981	0.215
$f^{\sharp}{}''$	254	76	900	987	0.210
f''	269	77	800	986	0.204
e''	285	78	700	986	0.199
$e^{\flat}{}''$	302	79	600	986	0.193
d''	320	80	500	987	0.188
$c^{\sharp}{}''$	339	81	400	987	0.182
c''	359	83	300	986	0.178
b'	380	84	200	984	0.172
$b^{\flat}{}'$	402	85	100	981	0.167
a'	425	86	0	977	0.161
$g^{\sharp}{}'$	449	88	-100	971	0.158
g'	474	89	-200	964	0.153
$f^{\sharp}{}'$	501	91	-300	960	0.149
f'	528	92	-400	950	0.144

Tab. 3: Ruckers 1644, spreadsheet part 1

A practical example

Now the reader should try to apply the mathematics explained up to now. A practical example is shown in the spreadsheet of Tab. 3. This spreadsheet calculation will be continued later in the article.

The data for the sounding lengths $\boldsymbol{L}$ of the string and the distances $\boldsymbol{P}$ from the nut to the plucking points are the original values measured at the 8' bridges of a historical instrument: harpsichord Andreas Ruckers 1644 in the Museum Vleeshuis at Antwerp/Belgium.

The instrument was originally tuned in meantone temperament or one of its variants. However, for the purpose of the calculations executed later in this article it is sufficient to use equal temperament where the intervals are simply multiples of 100 cents. The intervals $\boldsymbol{I}$ starting at the tuning tone a' are listed in the third column of the spreadsheet.

Now we want to calculate the tensile stress $\boldsymbol{S}$. We assume the following data:

- wire density ϱ: 7.85 g/cm^3
- frequency of the tuning tone a': 415 Hz (=cycles per second)

The tensile stress $\boldsymbol{S}$ is calculated by means of the following formula:

$$\left[\mathbf{S} = 4 \times 7,85 \times (415 \times 2^{\frac{\mathbf{I}}{1200}} \times \mathbf{L})^2 \times 10^{-9}\right]$$

The formula is put between brackets to indicate that is not a general formula but one that contains specific values, i. e. the values for the tuning frequency and the wire density. In this formula $\boldsymbol{S}$, $\boldsymbol{I}$ and $\boldsymbol{L}$ are printed in boldface, because they refer to the respective columns of the spreadsheet. It may be awkward to translate this formula into the specific formula language of a computer spreadsheet application. With a scientific pocket calculator the formula should not be difficult to use.

The values of the tensile stress are a bit on the high side. Probably at the time of the Ruckers the maximum tensile stress for the treble strings was somewhat lower, meaning that their instruments were tuned at a lower pitch. The popular tuning frequency of 415 Hz may therefore be doubted. However, many sorts of modern harpsichord wire can be stretched so far without problems so that the tuning frequency of 415 Hz is attainable.

In the last column the *plucking point factor* φ is calculated by means of the following formula:

$$\varphi = \frac{P}{L} \times \left(1 - \frac{P}{L}\right)$$

The values of the plucking point factor will be needed, when we continue the spreadsheet calculations later in the article.

The relative scaling of string gauges

Now we come to the main point: how should the string gauges be increased from the treble to the bass? For the moment let's leave absolute gauges aside and deal only with the relative scaling of string diameters.

The guitar rule

Concerning guitars Fletcher/Rossing have stated (see [2] page 212): "There appears to be some advantage in selecting string gauges in such a way that the tensions in all six strings will be nearly the same (Houtsma, 1975)". Can this rule principally be applied to harpsichords, too? There is indeed a publication, where this rule has been proposed for the harpsichord.

First we will analyze the consequences of this rule mathematically. For the initial deflection D of the string caused by plucking the following formula is applicable:

$$D = \frac{\varphi F L}{T}$$

where φ is the plucking point factor, F the plucking force, L the sounding length of the string and T the string tension (All these terms have already been introduced.)

As for the plucking force in the case of the harpsichord we can quote Fletcher/Rossing: "For satisfactory playing, it is necessary that the force exerted on the keys be constant over the keyboard compass, and this is essentially equivalent to constant plucking force F, since variation in the pivot position of the keys is limited."[2] Thus, applying the guitar rule that the tension should be constant, in the above formula F and T are constants and we can conclude that with a constant tension the deflection would be proportional to the product of the plucking point factor φ and the string length L. What are the consequences?

The plucking point factor φ generally varies between about 0.24 in the extreme treble and about 0.08 in the extreme bass, that is by a factor of about 3. The sounding length L of the string, however varies by a factor of about 9 – Italian harpsichords 12 – or more between treble and bass. Thus, if the tension is kept constant, the deflection D of the string will increase from the treble to the bass by a factor of at least 3 or – with Italian harpsichords– 4; by this factor the key dip would also increase. Such an instrument would not be easy to play.

Keeping the string tension constant is, therefore, not a practical rule. This rule may also be discarded for another reason.

In the treble octaves (for the 8' strings: $1\,^1/_2$ octaves with non-Italian harpsichords and $2\,^1/_2$ to $3\,^1/_2$ octaves with Italian harpsichords) a so-called *just scale* [15] is very closely observed, that is the product of the fundamental frequency f and the string length L is kept constant and the tensile stress S remains constant, too.

Under this provision the tension can only be kept constant by using the same string gauge throughout the range of just scaling. However, from historical stringing lists we know that the string gauges were changed also in this range. The stringing list of a harpsichord made by Bartolomeo Cristofori[4] is quite unequivocal: This instrument has a *just scale* over three and a half octaves and over this compass five string gauges are prescribed! The physical reason for this historical practice is simple: By increasing the string diameter the deflection of the string is reduced resulting in a cleaner sound, because long excursions of the string induce distortions.

Thus, the rule of keeping the tension constant has never been applied by historical harpsichord makers.

A tentative stringing rule

Fletcher/Rossing knew that the guitar rule is not applicable to the harpsichord, since they proposed a completely different rule for it (see [2] page 298). Their proposal is quite appealing in spite of some caveats. The calculation procedure is somewhat complicated and must therefore be dealt with in much detail.

Fletcher's/Rossing's considerations are based on the energy E stored in the string by plucking. This energy is mathematically defined as follows:

$$E = \frac{2\varphi F^2 L}{\pi d^2 S}$$

where F is the plucking force, d the diameter of the string, L the string length, S the tensile stress of the string and φ the plucking point factor.

However, we are interested in the string diameter d; the above formula must, therefore be written in

[4] dated 1722, now in the collection of Leipzig University, catalog number 84. The instrument has the traditional Italian disposition $2 \times 8'$.

a different form to solve d.

$$d = \sqrt{\frac{2\varphi F^2 L}{\pi E S}}$$

At this point Fletcher/Rossing propose to keep the energy E constant; the plucking force F can be considered constant, too (see the quotation from [2] in the preceding subsection).[5]

Now we get a problem. With E and F being constant and L and S definite values, we will get a continuous range of values for d. However, there will be only some string gauges available. For example, the formula may give a value of 0.21 mm for d, while the available string gauges next to this value are 0.20 mm and 0.22 mm. The solution is to introduce two terms for the string diameter:

- the term $\boldsymbol{d}$ for the calculated diameters covering a *continuous* range of values,
- the term $\boldsymbol{G}$ for the real string gauges available with *stepwise* values.

However, if we use string gauges G that differ a bit from the continuous value for the string diameter d, the energy E cannot be exactly constant any longer (assuming a constant plucking force F as before). Rather the energy E will vary slightly.

Now the reader may ask: How do we know the values for the energy E and the plucking force F. The answer is, since we will initially deal with a *relative* string gauge scaling, we need not know them; the only important thing is that they are constant. We can eliminate all the constants (F, E, 2, π) from the above formula and write the following proportional relation:

$$d \propto \sqrt{\frac{\varphi L}{S}}$$

The symbol $\propto$ for *proportional to* should be noted. In words: The continuous string diameter d is proportional to the root of the product of the plucking point factor φ and the string length L divided by the tensile stress S.[6] This is the tentative stringing rule.

For the root expression the term *diameter variable* – abbreviated V – will be used:

$$V = \sqrt{\frac{\varphi L}{S}}$$

	$\boldsymbol{V}$	$\boldsymbol{d}$ (mm)	$\boldsymbol{G}$ (mm)	$\boldsymbol{E}$ [dB]	$\boldsymbol{T}$ (N)
c'''	0.209	0.185	0.199	-0.6	30
b''	0.213	0.189	0.199	-0.5	30
$b^{\flat}{''}$	0.218	0.193	0.199	-0.3	30
a''	0.222	0.196	0.199	-0.1	30
$g^{\sharp}{''}$	0.226	0.200	0.199	0.0	30
g''	0.229	0.203	0.199	0.2	31
$f^{\sharp}{''}$	0.232	0.205	0.199	0.3	31
f''	0.236	0.209	0.199	0.4	31
e''	0.240	0.212	0.222	-0.4	38
$e^{\flat}{''}$	0.243	0.215	0.222	-0.3	38
d''	0.247	0.218	0.222	-0.2	38
$c^{\sharp}{''}$	0.250	0.221	0.222	0.0	38
c''	0.254	0.225	0.222	0.1	38
b'	0.258	0.228	0.222	0.2	38
$b^{\flat}{'}$	0.261	0.231	0.222	0.4	38
a'	0.265	0.234	0.222	0.5	38
$g^{\sharp}{'}$	0.270	0.239	0.248	-0.3	47
g'	0.274	0.242	0.248	-0.2	47
$f^{\sharp}{'}$	0.279	0.246	0.248	-0.1	46
f'	0.283	0.250	0.248	0.1	46

Tab. 4: Ruckers 1644, spreadsheet part 2

Applying the tentative rule

Now it is time to continue the spreadsheet example (Tab. 3) with the data of the Ruckers harpsichord of 1644. In Tab. 4 we have calculated the diameter variable V with the data of the preceding spreadsheet by using the above formula.

The values already look like real string diameters, but they are a bit on the high side. We simply divide them by an appropriate factor to get values of the *continuous diameter* $\boldsymbol{d}$ (next column) that

[5] On page 298 of [2] the formula for the energy contains the plucking ratio instead of the plucking point factor assuming that $p \ll 1$. With values of about 0.1 for p the difference between the plucking ratio and the plucking point factor is indeed negligible. However, if the values are nearing 0.5 the difference becomes rather big. For the deduction of the formula see annex A.

[6] Within the range of *just scaling* of the harpsichord string lengths the tensile stress remains constant. Therefore, according to this tentative rule the string diameter is proportional to $\sqrt{\varphi \times L}$ in this range.

seem reasonable for the chosen historical instrument. Thereafter we choose those available string gauges $\boldsymbol{G}$ (assuming string gauges of 0.199 mm, 0.222 mm and 0.248 mm) that are closest to the calculated values of $\boldsymbol{d}$.

The next column of Tab. 4 contains the relative energy levels in decibels. The reference value is the energy which would be stored in the string by using the calculated string diameters of the column d – under the prerequisites already described. Wherever the real string diameter G differs from d, the energy is different. Typically the variations of the energy levels are in the range ±0.5 dB; at the points where the string gauge is changed, this may add up to about 1 dB. The resulting variations of the sound pressure level produced are inaudible.

Finally the tension should be calculated. By summing up the tension of all strings we will get the overall tension which the structure of the instrument must withstand.

It is evident that the only calculation based on scientific considerations is the calculation of the diameter variable V. The subsequent computations are based on the harpsichord maker's experience. Nevertheless, the computations of Tab. 4 follow certain mathematical rules that are explained in the following subsection.

The mathematics of Tab. 4

How the values of the diameter variable $\boldsymbol{V}$ are calculated, has already been described. For computing the next columns we make the following *assumptions*:

- In the highest treble the instrument sounds best with the minimum string gauge available which is 0.199 mm.
- According to the historical stringing list by Claas Douwes [13] (see Tab. 6) this gauge is used for the upper eight strings of the 8' register.
- The next string gauges available are 0,222 mm and 0,248 mm.

According to these assumptions the column destined for the real string gauges $\boldsymbol{G}$ can be filled with the value 0.199 from c''' to f'' and with 0.222 at e'' without any computation.

The first computation is quite easy: We calculate the mean of the two values of $\boldsymbol{V}$ at f'' and e'' and do the same for the respective values of $\boldsymbol{G}$ and divide the former mean by the latter one. This calculation is shown in the following little table:

	$\boldsymbol{V}$	Mean		$\boldsymbol{G}$
f''	0.236	$\Rightarrow 0.238$	$0.2105 \Leftarrow$	0.199
e''	0.240			0.222
$\frac{0.238}{0.2105} = 1.1306$				

Now we divide the values of the diameter variable $\boldsymbol{V}$ by the result of this computation, thereby getting the values of the continuous string diameter $\boldsymbol{d}$:

$$\left[d = \frac{V}{1,1306}\right]$$

The formula is put between brackets to indicate that it is not a general formula, but one that contains a specific value.

The real string gauges $\boldsymbol{G}$ from $e^{\flat\prime\prime}$ on are chosen in such a way that they are closest to the respective values of $\boldsymbol{d}$.

After the string gauges have been chosen, the tension $\boldsymbol{T}$ can be calculated:

$$T = \frac{\pi G^2 S}{4}$$

The energy is inversely proportional to the square of the string diameter. Since we use the energy stored in the string at the calculated string diameter $\boldsymbol{d}$ as reference, the relative energy level is calculated by means of the following formula:

$$E[\text{dB}] = 10 \log \left(\frac{\frac{1}{G^2}}{\frac{1}{d^2}} \right) = 20 \log \frac{d}{G} \text{ decibels}$$

As already remarked the variations of the energy level are typically within ±0.5 dB. However, depending on how many strings of the same gauge are used in the highest treble, the energy level will be beyond these limits at the highest few notes; it may drop by several decibels, which is an indication that in this range the tentative stringing rule is no longer valid.

Caveats

It is not necessarily a deficiency of the tentative stringing rule that it determines only the *relative* string diameters. There is, however, an important caveat:

According to historical stringing practice the same gauge was used for a varying number of treble strings. The greater this number the later we start with the tentative stringing rule and the smaller will be the string diameter calculated for the bass strings.

The following examples of harpsichords, where string gauges are written on the wrestplank or the nut, show that the number (printed in boldface) of the treble strings down to the first change of string gauge varies:

- harpsichords with compass up to c''':

  Giusti 1679[7] : **11** for the 8' registers, **13** for the 4' register,

  Cristofori 1726[8]: **10** for the 8' register, **11** for the 4' register,

- harpsichords with compass up to f''':

  Taskin 1787[9]: **15** for the 8' registers, **16** for the 4' register,

  Kirckman 1755[10]: **18** for the 8' registers, **18** for the 4' register,

  Shudi 1782[11]: **18** for the 8' registers, **30** for the 4' register.

The numbers are greater for the instruments with a compass up the f''' than for those with c'''. Obviously the historical harpsichord makers did not count that number, but rather had a fixed note where to make the first change of string gauge. In the case of the Cristofori and Taskin instruments listed above this note was d'' for the 8' and $c^{\sharp\prime\prime}$ for the 4' registers. In the other instruments the change of string gauge occurs only one to three notes lower – with the exception of the 4' register of the Shudi instrument, where the change occurs a full octave lower.[12]

Comparing the 4' registers of the Shudi and the Kirckman it is interesting that in the bass both instruments arrive at about the same gauges, although the point of the first change of string gauge in the treble is drastically different. It is evident that in the Shudi after the first change the string gauges must increase faster than in the Kirckman and that, therefore, different stringing rules must be applied. The Kirckman is more typically of historical stringing practice in that the diameters of the 4' strings increase more slowly from the treble to the bass than those of 8' strings and require therefore a somewhat different stringing rule.

The tentative stringing rule, as we have introduced it, does not contain an optional parameter to adapt it to such cases. Such a parameter – we will call it *tuning parameter* – would be advantageous not only to tackle such problems, but also to make the stringing rule more flexible. Such a flexibility is indeed needed, since the basic premise of the rule – to keep the energy constant – is no guarantee for favorable results. This is another fundamental caveat for two reasons:

- A well-balanced sound from the bass to the treble does not depend on the energy alone, but also on the constructional principles of the instrument.

- The criterion *well-balanced* itself is a matter of taste (like e. g. the preferences for Boesendorfer or Steinway grand pianos).[13]

A tuning parameter for the tentative stringing rule will, therefore, fully rely on subjective judgment.

[7] Collection Tagliavini, Bologna. This instrument is one of the rare Italian harpsichords with a disposition of $2 \times 8'$, $1 \times 4'$. A very similar instrument by Giusti dated 1676 is in the collection of Leipzig University.

[8] Leipzig University. This instrument has the unique disposition 8', 4', 2'.

[9] Beurmann Collection, Hamburg

[10] Russel Collection, Edinburgh

[11] Victoria and Albert Museum, London

[12] The stringing list by Claas Douwes, which we use for the spreadsheet example, is somewhat different, too (point of change e''); however, since there is no mention of a 4' register, this stringing list was obviously meant for virginals (perhaps also for small harpsichords without a 4') of the Flemish type, whereas the instruments listed above are all large harpsichords.

[13] Loudspeaker tests are quite revealing in this respect: Many listeners prefer speakers that are objectively not well-balanced in that their frequency response rises considerably in the bass.

Tuning the tentative stringing rule

The tuning parameter must in some way modify the relative energy level E[dB].

Therefore the *tuning parameter* τ[dB] (the Greek letter tau), which is expressed in decibels, is introduced here. It is the increase (if positive) or decrease (if negative) of the relative energy level *per note* from the treble to the bass. A zero means a ratio of 1 : 1, that is no change.

For demonstrating the calculations with the *tuning parameter* we need to expand the spreadsheet example introduced with Tab. 3 and Tab. 4.

	$\boldsymbol{d}$ (mm)	$\boldsymbol{\tau}$ [dB]	$\boldsymbol{d_\tau}$ (mm)	$\boldsymbol{G}$ (mm)	$\boldsymbol{E}$ [dB]
f''	0.209	0	0.209	0.199	0.4
e''	0.212	0	0.212	0.222	-0.4
$e^{\flat\prime\prime}$	0.215	0.04	0.214	0.222	-0.3
d''	0.218	0.08	0.216	0.222	-0.2
$c^{\sharp\prime\prime}$	0.221	0.12	0.218	0.222	0.0
c''	0.225	0.16	0.221	0.222	0.1
b'	0.228	0.20	0.223	0.222	0.2
$b^{\flat\prime}$	0.231	0.24	0.225	0.222	0.4
a'	0.234	0.28	0.227	0.222	0.5
$g^{\sharp\prime}$	0.239	0.32	0.230	0.222	0.6
g'	0.242	0.36	0.232	0.222	0.8
$f^{\sharp\prime}$	0.246	0.40	0.235	0.248	-0.1
f'	0.250	0.44	0.238	0.248	0.1

Tab. 5: Ruckers 1644, spreadsheet part 3

In the expanded spreadsheet (Tab. 5) you will note two new columns: a column for the *tuning parameter* $\boldsymbol{\tau}$**[dB]**and a column for the *"tuned" continuous string diameter* $\boldsymbol{d_\tau}$.

The column $\boldsymbol{d}$ of this spreadsheet is identical to the respective column of Tab. 4; we have only omitted the upper treble notes before the first change of the string gauge, because these notes will not be affected by the *tuning parameter.* In the column $\boldsymbol{G}$ the values are are also taken from the respective column of Tab. 4, but only down to the first note after the first change of the string gauge – in the example down to the note e''.

Down to that note the column of the *tuning parameter* $\boldsymbol{\tau}$**[dB]** contains only zeros, because there is no change in this range.

Below that note we have to determine the values of the *tuning parameter* $\boldsymbol{\tau}$**[dB]**. This is a subjective choice! In the example of Tab. 5 we have *very arbitrarily* determined an increase per note of 0,04 dB – for demonstration purposes only.

By means of the values of τ[dB] the *"tuned" continuous string diameter* $\boldsymbol{d_\tau}$ can be calculated:

$$d_\tau = d \times 10^{\frac{-\tau}{20}}$$

The formula shows how decibels are converted back to ratios. The decibel value is divided by 10 or 20 (see the subsection on decibels which number to choose) and the result is taken as the exponent of 10. In the above formula the sign of τ is negative, because the energy stored in the string is *inversely* proportional to the string diameter.

The actually available string gauges $\boldsymbol{G}$ are then chosen in such a way that they are closest to the respective values of $\boldsymbol{d_\tau}$. The example of Tab. 5 demonstrates that the next change of the string gauge (0.248 mm) occurs two notes lower than in Tab. 4. The column $\boldsymbol{E}$, which is calculated by means of the same formula as in Tab. 4 shows how the relative energy level is increased.

The all-important question is, of course, how to determine the *tuning parameter* τ[dB]. The following procedure is proposed:

- First make the calculations without the *tuning parameter.*
- If you find the lowest bass strings to be too thick (or – rather unlikely – too thin), then try some tuning parameters at the lowest bass notes, until the result satisfies you.
- Then choose an appropriate value of the increase per note, so that its running sum from the treble down to the bass ends up with the parameter you have chosen in the previous step.

The validity and practical value of the stringing rule

Following historical stringing lists we must conclude that in the upper treble the stringing rule is not applicable and that the rule should be applied starting at the first change of the string gauge downwards. However, considering that we have introduced a subjective tuning parameter the question may arise, whether the tentative stringing rule has any validity at all.

Generally, the tuning parameter will be such to raise the energy level in the lowest bass of the 8' register by no more than 1 dB.[14] Such a change is rather subtle and does not really question the validity of the basic premise of the tentative stringing rule, namely that the energy should be constant (we had better say that the energy should be *roughly* constant, of course).[15] This premise, however, assumes instruments with well-balanced acoustic properties, an assumption which is certainly true for historical harpsichords, but may incidentally be questioned in the case of wing spinets and virginals.[16] See also the section "Special cases" later in this article.

What about the practical value of the stringing rule? The answer is twofold:
First, with the *relative energy level* expressed in decibels the harpsichord maker can *see* the physical consequences of his stringing list and need not rely solely on his ears.
Second, by applying the "tuned" stringing rule with a constant increase per note the harpsichord maker ensures that the averaged curve of the energy level has a constant slope.

The minimum string gauge

The calculations according to the stringing rule started at the *treble end* of the instrument. The physical reason for doing so will now be explained.

During a cycle of the vibration the tensile stress varies due to the slight change in the length of the string, which occurs twice during a cycle.[17] The difference between the maximum and the minimum tensile stress during a vibration will be called ΔS and can be calculated by means of the following formula:

$$\Delta S = \frac{pM}{4}\left(\frac{F}{T}\right)^2$$

where M is the modulus of elasticity of the wire and – as already introduced – p the plucking ratio, F the plucking force and T the tension of the string.[18]

If ΔS becomes too great, a distortion of the sound, which will be described later, will be heard [8], and the string behavior can no longer satisfactorily be described by the simple linear equations presented in this article, rather nonlinear behavior will be dominating.[19] Nevertheless, the above formula is sufficient to tell us, where the harpsichord is particularly sensitive to such distortions. Let's simplify the above formula a bit. The modulus of elasticity is a constant and the plucking force can be considered constant, too. Thus, the following proportional relation can be deduced from the formula:

$$\Delta S \propto \frac{p}{T^2}$$

[14] According to historical stringing lists the string gauge used in the upper treble of the 4' register was only one gauge number smaller than the one used in the 8' or even the same, whereas in the bass the 4' strings were considerably thinner than the 8' strings. Therefore, the rise of the energy level towards the bass was somewhat more prominent in the 4' register.

[15] The reader is reminded that the principles of the stringing rule and its basic premise are found in a book [2] written by two leadings scientists in the field of musical acoustics. A conclusion that they were basically wrong would have been quite astonishing indeed.

[16] With very few exceptions Italian polygonal virginals made in the 16th and the beginning of the 17th century have a compass from C/E to f'''. There have been some controversial discussions about their original pitch [16, 18, 19]. Since at the time when they were made no music required a treble end beyond c''', the additional notes up to f''' could only have been used for transposing. The question is, whether the transposing was done from low to normal pitch or from normal pitch to high pitch. In the first case they would have been strung with brass throughout the compass, in the second iron strings would have been used in the treble. For acoustical reasons the latter case seems more plausible, since in the first the instruments would have been definitely too small for a compass down to a C sounding GG in normal pitch. The existence of two-manual Italian harpsichords with the 4' register on the upper manual confirms a musical practice to play at high pitch. Some 17th century French harpsichords have this disposition, too.

[17] See annex C

[18] The formula assumes rigid end supports. However, in practice the supports are not totally rigid, rather the structure of the instrument will be induced to vibrate. The longitudinal vibrations have twice the frequency of the transverse ones and indeed a vibration component having this double frequency has been measured on a real instrument [11].

[19] It should be noted that the formulas presented in this article are approximations assuming very small excursions of the string, which is the normal case in a well regulated harpsichord. With large string displacements they are not valid anylonger.

In words: ΔS is proportional to the plucking ratio and inversely proportional to the square of the tension. The tension T is thus the most sensitive parameter, because it is squared.

ΔS is thus greatest, where the tension is lowest and this is in the upper treble of a harpsichord. Unfortunately, exactly in this range the second parameter, the plucking ratio p reaches its greatest values in a harpsichord. The conclusion is clear: The upper treble strings are the most sensitive ones to produce distortions.

If such distortions occur, the remedy is simple: Take thicker strings, since this will increase the tension and reduce the initial deflection of the string. On the other hand, if there is no audible distortion, try thinner strings.

It should be noted that taking thinner strings without changing the pitch is equivalent to tuning a string down i. e. lowering the pitch without changing the string. In both cases the tension is reduced. The effects of tuning a string down – i. e. the distortions just mentioned – have vividly been described by Frank Hubbard ([15] p. 9):

"If one tunes a harpsichord string down in pitch, listening carefully to the timbre as the pitch falls, a point will be reached where the tone becomes false and weak. Careful listening will identify the sensory cause of this effect. With the ictus at the pluck the pitch is high, and as the string sounds with attenuating volume the pitch falls slightly. This phenomenon is usually accompanied by beats so that the total effect of the tone is of a lack of homogeneity and force."

Now the reason for the *just scale* (i. e. the product of frequency and string length and thus the tensile stress is constant), which the historical harpsichord makers applied in the treble within rather close tolerances, becomes obvious: Within the range of *just scaling* the tensile stress was kept at a practical maximum, so that thin strings could be used without unduly reducing the tension.

From the formula just introduced

$$\Delta S = \frac{pM}{4}\left(\frac{F}{T}\right)^2$$

we can draw another conclusion: If we use wire with a lower modulus of elasticity (M), then the tension can be lower, too, to arrive at the same value for ΔS. The practical importance of this conclusion is that about the same gauges should be used for the brass strings of short-scaled Italian harpsichords as for the iron strings of Flemish and other long-scaled instruments in the treble, provided that their pitch is equal. The lower tension of the brass strings in the Italian harpsichords is roughly compensated for by their lower modulus of elasticity.

Can the minimum string gauge required for the harpsichord be calculated? Unfortunately not. The above formula only points at the critical range, where the harpsichord maker should find the string diameter that is best suited for the individual instrument – by trial and error. It is, however, well-known that the minimum string gauge for 8' strings is quite small: with 0.2 mm diameter in the treble audible distortions are very unlikely to occur. For 4' strings even smaller diameters can be used; the same applies to small high-pitched instruments, which are dealt with later in this article.

Nevertheless, the question is, whether increasing the string diameter has any acoustical advantage.

Using thicker strings?

What are the physical consequences, if the string diameter d is increased? We will first examine the impact on the energy E and the deflection D of the string.

If the sounding length L of the string, its tensile stress S and the plucking point factor φ are kept constant, then the following proportional relations are valid:

$$E \propto \frac{F^2}{d^2} \qquad D \propto \frac{F}{d^2} \qquad \text{if } L, S \text{ and } \varphi \text{ constant}$$

Thus, if the plucking force F is kept constant, too, then both the energy E and the deflection D are inversely proportional to the square of the string diameter d. If for example a string diameter of 0.2 mm is increased by a factor of $\sqrt{2}$ (1.414) to 0.282 mm, then the energy is halved, which is equivalent to a drop of the energy *level* by 3 dB. Although it is not easy to relate energy levels to sound pressure levels produced by the instrument, there can be no doubt that such a drop of the energy level will be perceived as a slight drop of loudness.

Besides the drop of loudness there will be two other negative consequences: the added strain – by increasing the string diameter by $\sqrt{2}$ the tension is doubled! – on the structure of the instrument and the increased inharmonicity of the thicker strings. There is only one positive effect: the decay time will be increased, since it "is proportional to the wire density and ... depends in a rather more complicated way on wire radius and on vibration frequency." [1] (The loss of energy during vibration is primarily due to viscous friction in the air, not to the loss through the bridges.)

The drop of the energy can *theoretically* be compensated for by increasing the plucking force by the same factor as used for increasing the string diameter, in the above example by the factor 1.414. This can be done by using stiffer plectra or by positioning the plectra nearer to the string – the latter solution will probably require lead weights in the jacks to avoid hangers. In practice this means increasing the force on the keys by about the same factor. Assuming that previously the required force on the keys was normal and not unusually soft, this would result in a very heavy touch, at least when playing with two or even three registers,[20] which very few harpsichord players would tolerate. What is worse, however, is that too strong a pluck is detrimental to the sound, i. e. the sharp transient at the moment of the pluck is emphasized and the tone becomes harsh and aggressive.

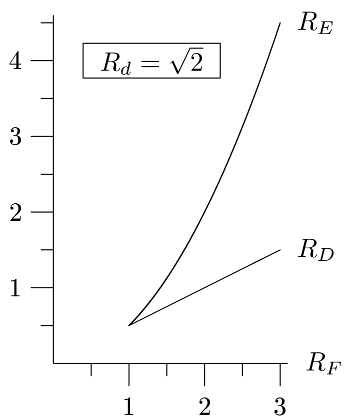
Increasing the plucking force considerably is, therefore, not practical with the delicate plucking action of the harpsichord, and it can be safely concluded that using much thicker strings (there are of course narrow subjective margins) than necessary for an undistorted sound (see preceding section) is not a viable concept for plucking keyboard instruments.[21]

The attentive reader may have noted that the energy is proportional to the *square* of the plucking force, whereas the deflection increases only in a linear manner. This is quite interesting and the key to understanding the concept of the harpsichord's successor, the pianoforte.

According to the proportional relations the following equations can be written:

$$R_E = \frac{R_F^2}{R_d^2} \qquad R_D = \frac{R_F}{R_d^2} \qquad \text{if } L, S \text{ and } \varphi \text{ constant}$$

where R_d and R_F are the ratios of a *chosen* string diameter and plucking force respectively divided by reference values and R_E and R_D the *resulting* energy and deflection ratios. Since in the equations there are tow independent variables (R_d and R_F), they can only be plotted in a three-dimensional graphic. For a two-dimensional graphic, we assume a fixed ratio $R_d = \sqrt{2} = 1.414$, that is the string diameter is increased by a factor of $\sqrt{2} = 1.414$ just like in the example above.



The physical law is evident: The curve of the energy ratios R_E is a parabola, whereas the deflection ratios R_D follow a straight line. By increasing the diameter ratio R_d both curves will be set lower and the slope of the parabola will increase more slowly.

It is this physical law (although the formulas of the plucked string are not exactly valid for the struck string) that enabled the makers of the pianoforte, who further developed Cristofori's invention, to make their instruments louder than the

[20] With two or more register switched on, the plucking processes start and end consecutively, but overlap partially, requiring a greater force on the keys than a single register.

[21] "Subjective loudness ...also depends upon decay time, since the brain tends to integrate sensation to some extent." ([1] p. 144) Since the decay time is proportional to wire density [1, 2], using brass instead of iron strings enhances subjective loudness a bit. This is certainly the reason why Italian harpsichord makers stuck to brass and also the German harpsichord maker Mietke used brass in the treble. However, with a compass up to f''' and a disposition of $2 \times 8'$, $1 \times 4'$ brass stringing is impossible because the 4' strings would be too short in the upper treble to accommodate three registers. Although the transposing harpsichords of the Ruckers merely had a compass up to c''', the four registers required more space than brass strings would have provided in the treble; so they were forced to use iron strings in the treble. Obviously the use of iron strings was not a philosophical question but simply a practical requirement.

harpsichord, since with the quite efficient piano action[22] a much greater energy could be transferred to the string and the string diameters had to be increased considerably to limit the deflection.

However, as stated before, with the delicate plucking action of the harpsichord this physical law cannot be exploited to increase the sound pressure level of the harpsichord. By making the whole instrument as resonant as possible, historical harpsichord makers attained a reasonable sound volume.[23]

Special cases

Stringing small high-pitched instruments

"Smaller and shorter scaled instruments were assigned thinner strings by Flemish and Dutch experts of the seventeenth and eighteenth centuries (see Douwes' stringing list ...)." Tab. 6 showing two stringing lists[24] by Claas Douwes confirms this statement by Frank Hubbard ([15] p. 9).

	6'-instr.		**5'-instr.**	
	gauge no.		gauge no.	
f'' to c'''	10	w	11	w
$b^{\flat\prime}$ to e''	9	w	10	w
$e^{\flat\prime}$ to a'	8	w	9	w
$g^{\sharp}$ to d'	7	w	8	w
d to g	6	w	7	w
B to $c^{\sharp}$	5	y	6	y
$A, B^{\flat}$	4	y	5	y
F, G	3	y	4	y
E	2	r	3	r
D			2	r
C	1	r	1	r

Tab. 6: Two stringing lists by Claas Douwes (1699)

This stringing practice can be confirmed physically. Simplifying the reality somewhat we assume that the small instrument is an accurately scaled-down version of a normal-size instrument, and that its pitch is higher by exactly the scaling factor. Then the product $f \times L$ and hence the tensile stress S as well as the plucking point factor φ are identical. The plucking force will also be the same and it is, of course, not desirable to reduce the energy. If we now take a look at the rewritten energy formula,

$$d = \sqrt{\frac{2\varphi F^2}{\pi E S}}\,\sqrt{L}$$

we can see that the left root expression contains just the parameters that are assumed to be identical between the scaled-down and the normal-size instrument. Thus, the string diameters are proportional to the root of the string lengths, which are shorter in the scaled-down instrument.

This is an oversimplification, of course, and such a rule cannot be applied exactly. However, the principle is clear and confirms historical stringing practice.

The Flemish *muselar*-virginal

Before dealing with string scales and stringing lists Claas Douwes [13] mentions the two different types of virginals used in the Low Countries in the 17th century: the *spinet*-virginal and the *muselar*-virginal. Obviously his stringing lists are meant to be applicable to both types.

Having the tentative stringing rule in mind this is somewhat surprising, since in the bass the *plucking point factor* φ of the *muselar*-virginal is about twice the one of the *spinet*-virginal and – according to the stringing rule – the strings should consequently be thicker by a factor of about one and a half.[25] Since Douwes' stringing lists seem to be rather realistic, we should try to find a physical explanation for this seeming incongruence.

It is well-known that the sound of these two types of virginals is drastically different; the sound of the *muselar*-virginal is somewhat hollow with emphasis on the fundamental, due to the plucking ratio, which is about 0.5 throughout the compass.[26] The

[22] The energy is defined as mass times the square of the velocity. It is in fact the velocity component of the energy that is so much greater in the pianoforte than in the harpsichord.

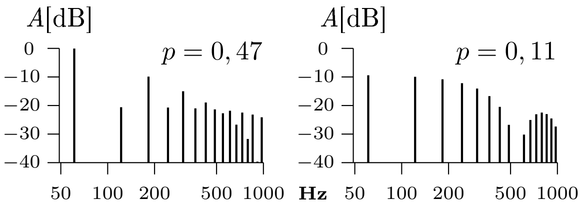
[23] Massive cases, heavy stringing and thick soundboards were the fundamental flaws of the "revival" harpsichords.

[24] Smaller gauge numbers mean thicker strings. The letters w, y and r indicate the colors of the string materials iron, yellow brass and red brass respectively. For the background of the gauge numbers see [21].

[25] In the *muselar*-virginal the strings are plucked near the middle throughout the compass, whereas the plucking ratios of the Flemish *spinet*-virginal are similar to those of a harpsichord.

[26] For a theoretical analysis see for example [10].

following figures show the theoretical (neglecting nonlinear string behavior) amplitude spectra transferred by the vibrating string to the bridge at the lowest bass note C; the left figure for the *muselar*-virginal and the right one for the *spinet*-virginal with assumed plucking ratios of 0.47 and 0.11 respectively.[27]



It can clearly be seen that the string of the *muselar*-virginal produces the stronger fundamental at the bridge. However, this does not relate to the sound produced by the instrument: At this low frequency the small soundboard of a virginal has very weak if any resonances to reinforce that frequency, so that the fundamental will hardly be heard. However, even if the fundamental is missing in the sound, the human ear can reconstruct it from the series of overtones.[28] The second harmonic is, therefore, much more important than the fundamental [2] and here the *spinet*-virginal is clearly stronger. The energy stored in the strings of both instruments being equal the *spinet*-virginal will be louder than the *muselar*-virginal in the bass.

This can be compensated for by enhancing the energy of the bass strings in the latter. That is exactly what is done by stringing the *muselar*-virginal like its sister instrument. However, since the deflection of the bass strings is thereby increased, a well-known effect – appearing if a long string is plucked in the middle – is reinforced, namely that the *muselar*-virginals "grunt in the bass like young pigs."[29]

[27]It should be noted that the maximum amplitude level is 0 dB and that the base line of the figures has *arbitrarily* been set to −40 dB; therefore the length of the bars is not proportional to the amplitude level. However, though a little bit misleading this sort of graphical representation is widely used and readers owning a digital audio recording machine with LED chains as level meters (mostly showing a minimum level of −60 dB) are already accustomed to it.

[28]See for example Zollner/Zwicker: *Elektroakustik*, 3. Aufl. Berlin/Heidelberg 1993, p. 20/21.

[29]Quirinus van Blankenburg, *Elementa Musica*, The Hague 1739, facs. Amsterdam 1972, extracts translated into English in [15], p. 237 and [19], p. 297.

Conclusion

The case of the *muselar*-virginal confirms the argument that there cannot be a general formula to calculate the string diameters. The individual acoustical properties of each instrument have to be taken into account and a stringing rule can only support the harpsichord maker but not replace his ears.

However, considering that three generations of modern harpsichord makers unsuccessfully tried to transfer concepts of piano making to the harpsichord, the ears do not seem to be reliable guides as long as the brain adheres to wrong concepts. With a little bit of physics the errors of these generations could have been avoided.

Since the publication of the books by Raymond Russel and Frank Hubbard [14, 15] the historical concepts of harpsichord construction are commonly known among modern makers. Nevertheless, looking at the few published modern stringing lists there still seems to be considerable uncertainty about stringing a harpsichord.[30]

Most of the physical foundations of harpsichord stringing described in this article have been known for one and a half centuries, but the flow of information from physicists to instrument makers is notoriously slow, since the former speak a language that the latter do not understand.[31] It is hoped that the reader found the language of this article understandable and will profit from the stringing rules presented here.

[30]A modern stringing list is published in [17], another one in the catalog of the Beurmann Collection at Hamburg, and a proposal for stringing comes with the drawing of the harpsichord Carlo Grimaldi, Messina 1697, which belongs to the collection of the *Germanisches Nationalmuseum*, Nuremberg. In the light of the stringing rules presented in this article all these three modern stringing lists seem to be questionable. – In contrast, the empirical stringing list developed by Frank Hubbard (see [15], p. 11) in the course of the restoration of an Italian harpsichord (Franciscus Marchionus 1666) confirms the stringing rules presented here. Only the use of steel instead of brass in the treble might be criticized, but thin brass strings were not available in the year 1949, when the restoration was made made.

[31]Seemingly some authors of the general literature on the harpsichord do not understand the language of the physicists either, otherwise there would not be the wrong statements about stringing occasionally found in this literature.

Annexes

The standard linear theory of the plucked string briefly summarized here is based on assumptions (e. g. rigid end supports, ideally flexible strings and linear string behavior) that differ more or less from real conditions.

The formula 42 has been developed by the author. Therefore its deduction – based on information in [2] page 209/210 – is described in full detail in annex C.

A Elasticity of the plucked string

A.1 Basic equations

$$M = \frac{S}{\varepsilon} \tag{1}$$

M : modulus of elasticity (Young's modulus)
S : tensile stress
ε : relative stretch

$$S = \frac{T}{A} = \frac{4T}{\pi d^2} \tag{2}$$

T : tension
A : cross-sectional area
d : wire diameter

$$\varepsilon = \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0} \tag{3}$$

L_0 : wire length – loose
L : wire length – stretched

From the equations 1 to 3 follows:

$$T = \frac{AM}{L_0}\Delta L \tag{4}$$

We recognize Hooke's law with the spring constant k of stretched wire:

$$k = \frac{AM}{L_0} \tag{5}$$

The general equation for the energy stored in a spring is:

$$E = \int_0^x kx\, dx = \frac{1}{2}\, kx^2 \tag{6}$$

The energy stored in a stretched piece of wire is therefore:

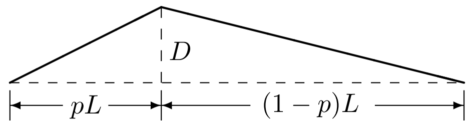
$$E = \frac{AM}{2L_0}(\Delta L)^2 \tag{7}$$

A.2 The energy stored in a plucked string

According to equation 4 the amount of stretch ΔL_s of a string is:

$$\Delta L_s = \frac{TL_0}{AM} \tag{8}$$

By plucking the string at the distance pL from a bridge[32] by a deflection D it will be stretched by an additional amount ΔL_p. The equation for ΔL_p can easily be deduced from the following figure (Pythagoras' law):



$$\Delta L_p = \sqrt{D^2 + p^2 L^2} + \\ + \sqrt{D^2 + (1-p)^2 L^2} - L \tag{9}$$

The total amount of stretch ΔL is therefore:

$$\Delta L = \Delta L_s + \Delta L_p = \\ = \frac{TL_0}{AM} + \sqrt{D^2 + p^2 L^2} + \\ + \sqrt{D^2 + (1-p)^2 L^2} - L \tag{10}$$

According to equation 7 the total energy E_{tot} in a plucked string is:

$$E_{tot} = \frac{AM}{2L_0}\left(\frac{TL_0}{AM} + \sqrt{D^2 + p^2 L^2} + \right. \\ \left. + \sqrt{D^2 + (1-p)^2 L^2} - L\right)^2 \tag{11}$$

The plucking force F is obtained according to Castigliano's law[33]

$$F = \frac{\partial E_{tot}}{\partial D} \tag{12}$$

i. e. by differentiating equation 11:

[32]The plucking ratio p is defined as $\frac{P}{L}$ where P is the distance from the plucking point to the *nearest* bridge and L the sounding length of the string. The plucking ratio is therefore limited as follows: $0 < p \leq 0,5$. The reason for this limitation is that the amplitude spectra of a vibrating string plucked at ratios p and $(1-p)$ are identical and no difference can be heard, if a string is plucked at a ratio of e.g. 40% or 60% of its sounding length.

[33]See: Hütte, *Die Grundlagen der Ingenieurwissenschaften*, 29. Auflage, Berlin 1991, p. E-90.

$$F = \frac{AM}{L_0}\left(\frac{TL_0}{AM} + \sqrt{D^2+p^2L^2} + \right.$$
$$\left. + \sqrt{D^2+(1-p)^2L^2} - L\right) \times$$
$$\times \left(\frac{D}{\sqrt{D^2+p^2L^2}} + \frac{D}{\sqrt{D^2+(1-p)^2L^2}}\right) \quad (13)$$

Since the deflection of a harpsichord string is very small, we may introduce the following approximations:

$$\sqrt{D^2+p^2L^2} \approx pL \quad (14)$$

$$\sqrt{D^2+(1-p)^2L^2} \approx (1-p)L \quad (15)$$

The equation 13 will thereby be much simplified, and since in a well regulated harpsichord the approximation is very accurate and we may use it as an equation:

$$F = \frac{T}{p\,(1-p)L}D \quad \text{or} \quad \boxed{F = \frac{T}{\varphi L}D} \quad (16)$$

$$\text{where}: \quad \varphi = p\,(1-p) \quad (17)$$

This is, of course, again Hooke's law. The spring constant of the plucked string is therefore:

$$k = \frac{T}{\varphi L} \quad (18)$$

The formula for the energy stored in the string by plucking can now easily be deduced:

$$E = \frac{T}{2\varphi L}D^2 \quad \text{or} \quad E = \frac{FD}{2} \quad (19)$$

Using the equations 16 and 2

$$D = \frac{\varphi FL}{T} \quad \text{and} \quad T = \frac{\pi d^2 S}{4}$$

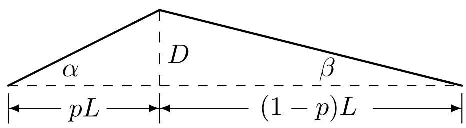
we obtain a more important form of this energy formula:

$$\boxed{E = \frac{2\varphi F^2 L}{\pi d^2 S}} \quad (20)$$

B Transverse vibration of the plucked string

B.1 The angles of the string at the bridges

The angles α and β of the plucked string relative to the horizontal line are shown in the following figure.



Since the angles are very small, the tangent function of the angle and the angle itself expressed in radians are practically identical.

$$\tan\alpha = \alpha_{(\mathrm{rad})} = \frac{D}{pL} \quad (21)$$

$$\tan\beta = \beta_{(\mathrm{rad})} = \frac{D}{(1-p)L} \quad (22)$$

$$\alpha_{(\mathrm{rad})} + \beta_{(\mathrm{rad})} = \frac{D}{p(1-p)L} = \frac{D}{\varphi L} = \frac{F}{T} \quad (23)$$

The sum of the angles α and β does therefore not depend on the plucking point.
From the equations 21 and 22 we obtain:

$$\boxed{\frac{\alpha}{\beta} = \frac{(1-p)}{p}} \quad (24)$$

B.2 The amplitude spectra of the plucked string

When stretched the string exerts a static downward force Q on the bridges. This transverse force periodically changes during vibration. The differences ΔQ between the actual and the static downward forces are proportional to the angles α and β [5] and thus, according to equation 24, as follows:

$$\Delta Q_\alpha \propto 1-p \quad (25)$$

$$\Delta Q_\beta \propto p \quad (26)$$

The total amplitude of these force differences $(1-p+p=1)$ does therefore not depend on the plucking point [4].
For angles above the base line ΔQ is negative (reduced downward force), for angles below the base line ΔQ is positive (increased downward force).

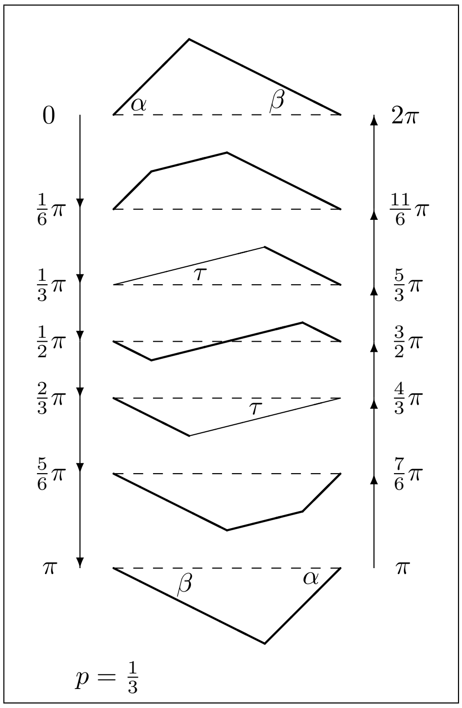
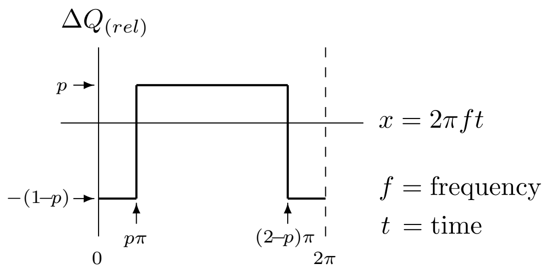


Fig. 1: Motion of a string

The motion of a string plucked one third ($p = 1/3$) of the distance from one bridge is schematically shown in Fig. 1. The moments where the angles turn from above the base line to below and vice versa are shown in the third drawing (left bridge: at $p\pi$ and $(2-p)\pi$) and fifth drawing (right bridge: at $(1-p)\pi$ and $(1+p)\pi$). The angle τ is transient at the moment of turn and does not affect the force on the bridges. (At the moment of turn, the string, which is periodically stretched during the vibration, has its minimum length – see annex C).

At the bridge nearest to the plucking point the string motion induces the *relative* differences $\Delta Q_{(rel)}$ between the static and the actual downward forces as shown in the following figure.



By Fourier analysis of this square wave we get:

$$\Delta Q_{(rel)} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin(np\pi)\cos(nx)\right] \tag{27}$$

where n is the nth number of the harmonic.[34]

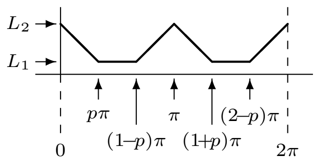
Thus, the the amplitude spectrum of the square wave is

$$\boxed{A_n[\mathrm{dB}] = 20\log\left|\frac{1}{n}\sin(np\pi)\right|} \tag{28}$$

where the reference value is 1, i. e. the greatest possible amplitude reached at $n = 1$ and $p = 0,5$.[35]

C The change in the longitudinal force

During a cycle of the vibration the longitudinal force changes due to the slight change in the length, which occurs twice during a cycle.[36] With $p = 1/3$ this is shown in the figure below.



L_2 is the maximum and L_1 the minimum string length during a cycle.. The base line in the figure refers to the length L of the string at rest. If $p = 0,5$ then $L_1 = L$.

As already shown in subsection A.2 the maximum length L_2 will be reached at the initial deflection D.

$$L_2 = \sqrt{p^2L^2 + D^2} + \sqrt{(1-p)^2L^2 + D^2} \tag{29}$$

According to equation 16 the deflection D is (assuming very small deflections):

$$D = \frac{p(1-p)LF}{T} \tag{30}$$

[34] At the other bridge we get a rotationally symmetrical square wave. The Fourier series is, therefore, the same apart from the sign of n, which is different for even and odd n's.

[35] The sign of the sine values may be neglected, because the human ear is insensitive to phase differences of overtones.

[36] see measurements in [4].

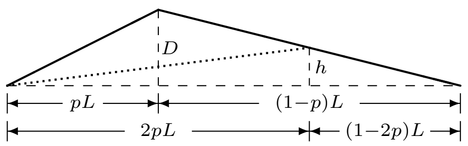
Thus we get:

$$L_2 = pL\sqrt{1+\frac{(1-p)^2F^2}{T^2}} + + (1-p)L\sqrt{1+\frac{p^2F^2}{T^2}} \qquad (31)$$

Since the values of the fractions in the root expressions are very small, the following very accurate approximation (we use the symbol $=$ instead of $\approx$) is valid:

$$\begin{aligned} L_2 &= pL\left(1+\frac{(1-p)^2F^2}{2T^2}\right) + \\ &\quad + (1-p)L\left(1+\frac{p^2F^2}{2T^2}\right) \\ &= L + \frac{Lp(1-p)F^2}{2T^2} \end{aligned} \qquad (32)$$

The minimum string length L_1 is reached at the moment, where the angle of the string turns from above the base line to below (see subsection B.2); the dotted line in the following figure shows the string at that very moment:



From the figure we can deduce the formula for L_1:

$$L_1 = \sqrt{4p^2L^2+h^2} + + \sqrt{(1-2p)^2L^2+h^2} \qquad (33)$$

The momentary deflection h is given by:

$$\frac{h}{D} = \frac{1-2p}{1-p} \qquad (34)$$

$$h = \frac{p(1-2p)LF}{T} \qquad (35)$$

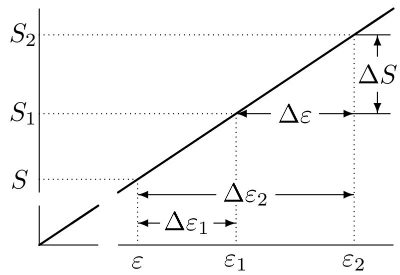
Now for L_1 we get:

$$L_1 = 2pL\sqrt{1+\frac{(1-2p)^2F^2}{4T^2}} + + (1-2p)L\sqrt{1+\frac{p^2F^2}{T^2}} \qquad (36)$$

Like in the equation 32 the following very accurate approximation is valid:

$$\begin{aligned} L_1 &= 2pL\left(1+\frac{(1-2p)^2F^2}{8T^2}\right) + \\ &\quad + (1-2p)L\left(1+\frac{p^2F^2}{2T^2}\right) \\ &= L + \frac{Lp(1-2p)F^2}{4T^2} \end{aligned} \qquad (37)$$

Per definition L is the stretched string when it is at rest, S its tensile stress and ε the relative amount of stretch at this state. Analogously S_1 and ε_1 refer to L_1, whereas S_2 and ε_2 refer to L_2. This is shown schematically in the figure below:



Assuming Hooke's law still to be valid (i. e. the string is not stretched beyond the proportional range) we get (see equation 1):

$$M = \frac{S}{\varepsilon} = \frac{\Delta S}{\Delta \varepsilon} \quad \rightarrow \quad \Delta S = M\,\Delta\varepsilon \qquad (38)$$

$\Delta\varepsilon_2$, $\Delta\varepsilon_1$ and $\Delta\varepsilon$ are given by:

$$\Delta\varepsilon_2 = \frac{L_2 - L}{L} = \frac{p(1-p)F^2}{2T^2} \qquad (39)$$

$$\Delta\varepsilon_1 = \frac{L_1 - L}{L} = \frac{p(1-2p)F^2}{4T^2} \qquad (40)$$

$$\Delta\varepsilon = \Delta\varepsilon_2 - \Delta\varepsilon_1 = \frac{pF^2}{4T^2} \qquad (41)$$

Now the change ΔS of the tensile stress occurring during a cycle of the vibration can be determined:

$$\boxed{\Delta S = \frac{pM}{4}\left(\frac{F}{T}\right)^2} \qquad (42)$$

References

[1] Neville H. Fletcher, "Analysis of the Design and Performance of Harpsichords" , in *Acustica* 37 (1977), p. 139

[2] Neville H. Fletcher and Thomas D. Rossing, *The Physics of Musical Instruments*, New York 1991; especially chapter "Harps, Harpsichords, and Clavichords"

[3] C. Cuesta and C. Valette, "Evolution temporelle de la vibration des cordes de clavecin" in *Acustica* 66 (1988), p. 37

[4] C. Cuesta and C. Valette, "Le transitoire d'attaque des cordes de clavecin" in *Acustica* 68 (1989), p. 112

[5] Lothar Cremer, *Physik der Geige*, Stuttgart 1981 (English translation available: *The Physics of the Violin*, Cambridge, Massachusetts 1984), especially Chapter 2 "Die gezupfte Saite" ("The Plucked String")

[6] K. A. Legge and N. H. Fletcher, "Nonlinear generation of missing modes on a vibrating string" in *Journal of the Acoustical Society of America* 76 (1), July 1984, p. 5

[7] C. Karp, "The Inharmonicity of Strung Keyboard Instruments" in *Acustica* 60 (1986), p. 295

[8] E. Lieber, "Moderne Theorien über die Physik der schwingenden Saite und ihre Bedeutung für die musikalische Akustik" in *Acustica* 33 (1975), p. 324

[9] C. Cuesta and C. Valette, " Théorie de la corde pincée en approximation linéaire" in *Acustica* 71 (1990), p. 28

[10] Nicolas Meeus, "Epinettes et 'muselars': une analyse théorique" in *La facture de clavecin du XV^e au XVIII^e siècle* (Editor: Philippe Mercier), Louvain-la-Neuve, 1980

[11] Herbert A. Kellner, "Über den Schwingungsmechanismus beim Cembalo" in *Das Musikinstrument*, volume 23 (1974), p. 187-192

[12] Bram Gätjen, *Der Klang des Cembalos*, Kassel 1995

[13] Claas Douwes *Grondig Ondersoek van de Toonen der Musijk*, Franeker, 1699 (facs. Amsterdam 1970). Extracts translated into English in [15] and [19].

[14] Raymond Russel, *The Harpsichord and Clavichord*, Second, revised edition, London 1973 (First edition 1959)

[15] Frank Hubbard, *Three Centuries of Harpsichord Making*, Cambridge, Massachusetts 1965

[16] John Barnes, "The Specious Uniformity of Italian Harpsichords" in *Keyboard Instruments*, edited by Edwin M. Ripin, New York 1977

[17] Hubert Henkel, *Beiträge zum historischen Cembalobau*, Leipzig 1979

[18] Edwin M. Ripin et al. *Early Keyboard Instruments* (The New Grove Dictionary of Musical Instruments), New York and London 1989

[19] Grant O'Brien, *Ruckers – A harpsichord and virginal building tradition*, Cambridge 1990

[20] John Henry van der Meer, "Die Geschichte der Zupfklaviere bis 1800" in *Kielklaviere – Cembali, Spinette, Virginale* (Catalog of the Berlin Collection), Berlin 1991

[21] Klaus Scholl, *Historische Saitendrahtnummern mathematisch betrachtet*, published by Marc Vogel oHG, 2007 (www.vogel-scheer.de)

The references [1] to [12] dealing with *physics* are listed in order of relevance to the subject of this article, references [13] to [21] dealing with *historical harpsichord making* in chronological order.